Numberphile Seminar
WS19/20
Universität Freiburg

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Übersicht

Thema

Es gibt im Internet eine Reihe interessanter Videos zu mathematischen Themen. Beispielsweise den YouTube-Kanal Numberphile rund um Brady Haran, aber noch eine Reihe weitere. Dort wird ganz verschiedenartige Mathematik präsentiert: Etliches kann man vielleicht als "Unterhaltungsmathematik" bezeichnen, aber viele Videos greifen auch sehr nichttriviale Mathematik auf und bemühen sich, sie einem größerem Publikum verständlich zu erklären - also insbesondere einem Publikum, was nicht gerade das Wissen aus mehreren Semestern Mathematikstudium zur Hand hat.

Aber das ist bei uns ja anders. Wir kennen ein bisschen mehr Mathematik und wir können uns die Themen, die hinter so manchen Videos stecken, auch etwas genauer anschauen.

Der Plan des Seminars ist daher, zu einzelnen Videos ein wenig die "Hintergrundstory" zu erkunden. Um nur ein Beispiel zu nennen: Die Fibonacci-Folge ist eine ziemlich berühmte Folge. Auch etliche Menschen, die keine Mathematik studiert haben, haben zumindest den Begriff schon mal gehört. Auch auf Numberphile spielt diese Folge eine Rolle, in gleich mehreren Videos. Allerdings kann man sich solche Folgen auch von einem viel allgemeineren Standpunkt anschauen: Es ist ein Spezialfall einer sogenannten "linear rekurrenten Folge". Zu diesen gibt es eine ganze Theorie, die übrigens alles andere als trivial ist. Und die in der Unterhaltungsmathematik wohlbekannte Tatsache, dass es eine geschlossene Formel für die n-te Fibonacci Zahl gibt (Binets Formel) ist von der allgemeinen Theorie her wenig überraschend: es gibt solch eine Formel für jede lineare rekurrente Folge. Man kann die n-te Fibonacci-Zahl aber auch berechnen, indem man die n-te Potenz des goldenen Schnitts nimmt (durch Wurzel fünf teilt) und dann rundet. Das kann man ganz elementar zeigen. Aber auch dies ist nur eine Facette der allgemeineren Theorie und hat damit zu tun, dass der goldene Schnitt eine sogenannte Pisot-Zahl ist. Von dort ist man nicht weit vor der algebraischen Zahlentheorie, oder aber auch der Lehmer Vermutung (eine immer noch offene Vermutung). Und natürlich hat das auch alles was mit Kunst zu tun, und hat man nicht schonmal wo gehört, das Fibonacci-Zahlen was mit Sonnenblumen zu tun haben?

Dies ist aber nur ein Teilaspekt. Es gibt Numberphile Videos auch zu vollkommen anderen Themen aus der Mathematik (z.B. Fraktale, Kartentricks oder Würfeln) und ich hoffe, dass wir insgesamt eine gewisse Balance finden können, so dass für jeden Geschmack etwas dabei ist.

Format: Vorträge sollen nur 80 min lang sein, damit wir danach noch etwas gemeinsam diskutieren können. Es gibt keine "fertige" Literatur, dazu ist unser Seminarthema etwas zu unklassisch. Die Quellenlage wird meist so sein: (a) es gibt ein Numberphile Video, was allenfalls die Oberfläche ankratzt, (b) es gibt Fachliteratur, die Themen systematisch entwickelt, aber viel zu systematisch und länglich als für uns relevant. Die Kern-Aufgabe für Vortragende ist daher, einen guten Mittelweg zu finden. Das ist eine anspruchsvolle Aufgabe.

Talks

24.10.2019: Chaos Game/Iterated Function Systems (Deitmar)

`https://www.youtube.com/watch?v=kbKtFN71Lfs`

Explain contraction mapping principles and iterated function systems. Perhaps get inspiration from

- [Barnsley], Fractals Everywhere, Chapter IX.
- Nonlinear Dynamics And Chaos: Proceedings Of The Fourth Physics Summer School, Lecture 4, especially the Collage Theorem and Elton's Theorem (Palmer, Kenneth J., Bifurcations, chaos and fractals. Nonlinear dynamics and chaos (Canberra, 1991),)

The key point to understand what happens is Elton's Theorem or a variant thereof, so this should be the central result of the talk. The original proof is too hard. Perhaps [Forte, Mendivil - A classical ergodic property for IFS: a simple proof] is a better source? The main point is to make the statement believable and prove it with a reasonable choice of rigour given the audience.

31.10.2019: Measuring Coastline (Tobian)

`https://www.youtube.com/watch?v=7dcDuVyzb8Y`

Explain the original idea a bit further. Introduce the concept of fractal dimension (box dimension, fractal dimension, there are various variants...), prove their comparison, zeroness on countable sets. Compute these dimensions for various examples (see Mark Pollicott, Lecture Notes on Fractrals and Dimension, Porto Lectures).

Page 1-11 of Pollicott's Lecture Notes already cover most of this.

A key point in the coastline paradox is that while "finer" coastline data adjusts the area only a bit, the perimeter can explode. This links to the fact that the shortest path between two points is the straight line and anything else will be longer. One can give a broader outlook on the relation between area and perimeter (e.g. isoperimetric inequalities). If you want to provide further context, note also that there are geometric shapes like Gabriel's Horn.

One could also relate the idea of "fractal dimension" to other concepts of dimension, e.g. starting from linear algebra. Other concepts are for example the Lebesgue covering dimension.

Regarding fractal structures, there is a link to the talk "Chaos Game"; ideally exchange some ideas with the speaker for that talk beforehand.

7.11.2019: (fällt aus, Vortrag wurde abgesagt)

The talk was cancelled.

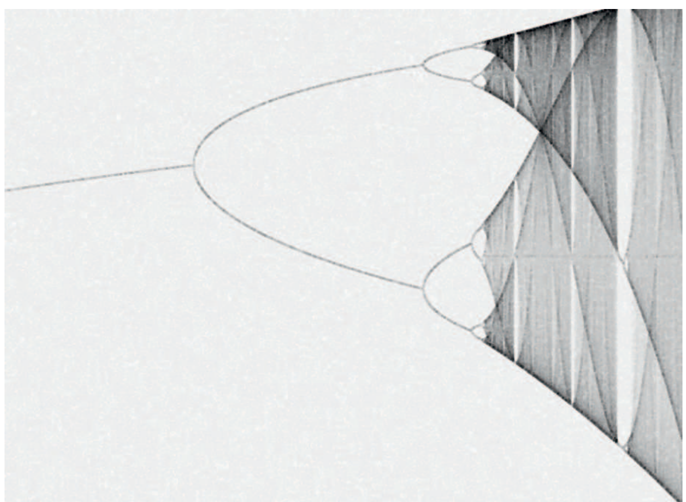
14.11.2019: The Feigenbaum Constant (4.669) - Numberphile (Moosmann)

`https://www.youtube.com/watch?v=ETrYE4MdoLQ`

Explain the logistic equation. If you want, you can elaborate a little more on "real world applications" of such ideas, e.g., in population models in biology, compare this to exponential growth etc.

Discuss the behaviour of the model when changing the reproduction factor/birth rate. There are various "standard ranges" where the behaviour of the model is easy to understand (e.g. population decays off to zero, oscillates between two values, between four values...) Explain how this change in behaviour arises (we can discuss this a little in the office hours if you have questions), and how to determine the values between which it oscillates.

Explain the famous picture:



The big question then is how Feigenbaum's constant turns out to be universal instead of just being bound to specific models. Discuss a bit the ideas of Feigenbaum, "Quantitative Universality for a Class of Nonlinear Transformations" or Lanford III "A computer-assisted proof of the Feigenbaum conjectures", or Eckmann-Wittwer "A complete proof....". The rigorous proof is certainly *far out of reach for us.* So, make an effort to explain the ideas a little bit without getting lost in the delicate details.

21.11.2019: Transcendental numbers (Brunn)

`https://www.youtube.com/watch?v=seUU2bZtfgM`

Explain $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\overline{\mathbb{Q}}$. Prove that the algebraic numbers $\overline{\mathbb{Q}}$ are a field. Prove that there are only countably many algebraic numbers, yet uncountably many real (or complex) numbers.

Prove that e or π (or both) are transcendental. The book [Delahaye] "Pi - Die Story" (also available in French) contains proofs for both, plus a lot of nice stories around transcendence. If you want a nice narrative and easy-going talk, this is a nice direction.

If you prefer to give a mathematically more challenging talk, you could sketch the proof of the stronger Hermite–Lindemann Theorem: If α is a non-zero algebraic number, then e^α is transcendental.

This result is *much* stronger: For $\alpha = 1$ it implies that e is transcendental immediately, and for $\alpha = i\pi$ it implies that π is transcendental (Proof: If π is algebraic, then so is $i\pi$ because i is algebraic and $\overline{\mathbb{Q}}$ a field. Thus, by Hermite–Lindemann $e^{i\pi} = -1$ is transcendental. This is a contradiction, because -1 is clearly not transcendental. Thus, π must have been transcendental all along). A complete proof of Hermite–Lindemann is given in [Lang, Serge] Introduction to Transcendental Numbers, Chapter III, §1 as "Corollary 1" to "Theorem 1". You would need to sketch §1-§3 in this chapter, plus sketch Siegel's Lemma. The proof is too long to present in full, so you would have to find a reasonable way to shorten it.

Note: The talk on linear recurrence sequences will briefly relate to the concept of an algebraic number or algebraic integer, so perhaps feel free to discuss a bit with the person giving that talk before your talks.

28.11.2019: Dehn invariant (Klingler)

`https://www.youtube.com/watch?v=eYfpSAxGakI`

Excellent source: Aigner, Ziegler - Proofs from THE BOOK, Chapter 8

1. Also report on the Wallace–Bolyai–Gerwien theorem, one can even reasonably sketch the proof.

2. Feel free to explain the tensor product (e.g. Eisenbud or Atiyah–MacDonald's books on commutative algebra discuss these in far broader detail, much more broadly than needed, perhaps the example $\mathbb{Z}/2 \otimes \mathbb{Z}/3 \cong 0$ is instructive)

3. Sydler has shown that scissor congruence holds if and only if volume and Dehn invariant match. If you want, report a little on the generalized problems in spherical or hyperbolic geometry.

There are various sources for additional material, e.g. Boltianskii's book on Hilbert's Third Problem (just browse a bit through it; perhaps the discussion of the area measurement problem at the beginning admits a vague thematic link to the talk about the Coastline problem?), Sydler's paper (although this approach is considered outdated), the Introduction on Dupont's book "Scissor congruences, group homology..." is also useful, although it very quickly moves into too advanced territory.

5.12.2019: Fantastic Quaternions: Part I, Algebra (Vollprecht)

`https://www.youtube.com/watch?v=3BR8tK-LuB0`

Introduce the quaternions. Explain the terms "scalar part", "vector part" and explain how to express the scalar product ("dot product") and the cross product $\times$ in terms of quaternions. Perhaps, even though this is high school maths, remind us of the geometric interpretation of scalar and cross product.

Then prove Frobenius' Theorem: Every finite-dimensional real division algebra is $\mathbb{R}$, $\mathbb{C}$ or Hamilton's quaternions $\mathbb{H}$; there are no others. Give an algebraic proof of this result.

Another interesting fact: The tensor algebra of two copies of the quaternions is isomorphic to the ring of matrices of the corresponding dimension, namely there is an isomorphism of rings

$$\mathbb{H} \otimes \mathbb{H} \simeq M_4(\mathbb{R}),$$

where $M_n(\mathbb{R})$ denotes the ring of $(n \times n)$-matrices with real entries (this is a special case of [Gille, Szamuely], Central Simple Algebra and Galois Cohomology, §1.5., Corollary 1.5.3; using that $\mathbb{H} = (-1, -1)$. There might be an easier sounce for this.

This talk is connected to Part II: Ideally communicate with the person giving the other talk beforehand to create a smooth transition between the talks.

12.12.2019: Fantastic Quaternions: Part II, Around the Brauer group (Landes)

`[same video as previous talk]`

Sketch the more general theory of central simple algebras (these generalize the quaternions). A quick treatment is found in [Gille, Szamuely], Central Simple Algebras and Galois Cohomology, Chapter 2 "Central simple algebras and Galois descent". A reasonable path:

- define central simple algebras as in §2.1,
- note that every division algebra gives rise to examples (Example 2.1.2)
- sketch a proof of Wedderburn's Structure Theorem, Theorem 2.1.3
- then explain the alternative characterization of central simple algebras given in Theorem 2.2.1 (again, only a sketch)
- (the material Prop. 2.2.5-2.2.6 just strengthens this to know that a Galois extension is enough; feel free to drop this)
- if you want, discuss without ANY PROOFS the concept of the Brauer group, §2.4; one can interpret

$$\mathbb{H} \otimes \mathbb{H} \simeq M_4(\mathbb{R})$$

as the statement that the Hamilton quaternions generate the 2-torsion group $Br(\mathbb{R}) = \mathbb{Z}/2$.

- also possible: Prove that every finite division algebra must be a finite field (Proofs from THE BOOK).

There will not be enough time to discuss all of these things, so you will have to shorten quite a bit.

This talk is connected ot Part I: Ideally communicate with the person giving the other talk beforehand to create a smooth transition between the talks.

19.12.2019: $1 + 2 + 3 + \ldots = -\frac{1}{12}$ (Paping)

`https://www.youtube.com/watch?v=w-I6XTVZXww`

Explain all the *serious* problems with this video and what's correct and what isn't. Remind the audience of concepts like absolute, unconditional or conditional convergence; especially the result that by suitably rearranging a conditionally convergent series, any limit is possible (there is also a generalization to $\mathbb{R}^n$, Lévy-Steinitz theorem)

You can talk about other summation methods for divergent series (e.g., Césaro summation) or the analytic continuation of the Riemann zeta function, which plays a decisive role for the question tackled in the video.

One can also take a more general high-brow viewpoint, as explained in Tao's blog
https://terrytao.wordpress.com/2017/05/11/generalisations-of-the-limit-functional/ ;

this also discusses ”Banach limits”, a very abstract concept generalizing the idea of a limit of a sequence.

9.1.2020: Fundamental Theorem of Algebra (Azarm)

[requires a little Galois theory]

https://www.youtube.com/watch?v=shEk8sz1oOw

Prove the Fundamental Theorem of Algebra quickly using Liouville's Theorem in complex analysis (e.g. as in Jänich, Funktionentheorie or any other book on complex analysis). Then show the alternative proof based on the intermediate value theorem of the reals and Galois theory (e.g. Milne, Theorem 5.6 in his lecture notes ”Fields and Galois Theory). Then prove the Artin–Schreier Theorem: If F is an algebraically closed field with some subfield R such that F/R is a finite field extension, then one must have $[F : R] = 2$ and $F = R[\sqrt{-1}]$. (for example follow Keith Conrad's proof).

16.1.2020: Lucas Numbers versus Fibonacci Numbers (Eger)

https://www.youtube.com/watch?v=PeUbRXnbmms

(if you want, feel free to address the weird concept of what makes a sequence ”better” (what would Karl Popper say?))

Explain the basic results of the theory of linear recurrences, especially the link: A sequence of numbers satisfies a linear recurrence if and only if its generating function is rational if and only if it is given through a generalized power sum.

Explain how the Fibonacci and Lucas numbers fit into this pattern. Explain what a characteristic root is and how this explains the asymptotics of various sequences. Explain why integer sequences correspond to characteristic roots which are *algebraic integers*; and perhaps what a Pisot number is.

The concept of an algebraic integer links back to the talk ”Transcendental numbers” earlier in the seminar (they are special elements of the field $\overline{\mathbb{Q}}$ and we will again see the minimal polynomial) and we'll see a little bit of Galois theory in action.

In particular, note that the Golden Ratio is an algebraic number.

Useful sources:

- the beginning of the little article [Rumely, van der Poorten] Remarks on generalized power sums,

and the book [Everest, van der Poorten, Shparlinski, Ward] Recurrence sequences.

We can further discuss in the office hours if something remains unclear.

Both sources cover MUCH more than what you should discuss, so don't feel intimidated by all the material in them.

23.1.2020

The meeting is cancelled.

30.1.2019: Chinese Remainder Theorem and Cards - Numberphile (Oelschlegel)

`https://www.youtube.com/watch?v=l9dXo5f3zDc`

Explain the Card tricks. Unlike the other talks, it's surely okay if you repeat the content of the video a bit more than in other talks, to help explaining what's going on. Prove the Chinese Remainder Theorem (CRT) for arbitrary commutative rings and pairwise coprime ideals. For this card trick, one uses the CRT to solve a *linear* system of congruences. This is indeed quite simple.

Now introduce the Chevalley–Warning theorem which gives an existence result for solving certain nonlinear equation systems in finite fields by a beautiful counting argument. Explain the concept of a quasi-algebraically closed field (a.k.a C_1-field).

There are many texts explaining Chevalley-Warning. The proof is a beautiful tricky counting argument, and quite fun in itself. If you want, you can say a bit more about existence theorems for solutions of certain equations.